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Calabi–Yau 4-folds and toric fibrations

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Abstract

We present a general scheme for identifying fibrations in the framework of toric geometry and provide a large list of weights for Calabi–Yau 4-folds. We find 914 164 weights with degree $d \leq 150$ whose maximal Newton polyhedra are reflexive and 525 572 weights with degree $d \leq 4000$ that give rise to weighted projective spaces such that the polynomial defining a hypersurface of trivial canonical class is transversal. We compute all Hodge numbers, using Batyrev's formulas (derived by toric methods) for the first and Vafa's fomulas (obtained by counting of Ramond ground states in $N = 2$ LG models) for the latter class, checking their consistency for the 109 308 weights in the overlap. Fibrations of k-folds, including the elliptic case, manifest themselves in the N lattice in the following simple way: The polyhedron corresponding to the fiber is a subpolyhedron of that corresponding to the k-fold, whereas the fan determining the base is a linear projection of the fan corresponding to the k-fold.

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1. Introduction

It used to seem obvious from the critical dimension of superstrings and the apparent dimension of space–time that only complex manifolds with dimensions up to 3 play a role in string theory. The second string revolution, however, has changed this picture. It was

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shown that strong coupling phenomena may increase the effective dimension of space–time to 11 dimensions [1], thereby providing a geometrical interpretation of a number of string dualities in the context of M-theory. Geometrization of the $SL(2, \mathbb{Z})$ symmetry of type IIB strings may even lead to 12 dimensions [2]. Whether or not there are situations with an effectively 12-dimensional space–time, F-theory compactification already has an impressive record as a way to talk about compactifications of type IIB strings and providing the missing geometrizations of duality symmetries [2–5]. In a related development, non-perturbative physics of three-dimensional compactifications seems to have the potential to teach us a lot about issues like SUSY breaking in four dimensions [4], which we may recover in some decompactification limit.

For all these issues it is important to have a number of generic examples of Calabi–Yau 4-folds at one's disposal and, in particular for applications in F-theory, to know how to identify elliptic fibrations in an easy way. The purpose of the present paper is twofold: On the one hand, we provide large classes of 4-folds in a systematic way. In addition we give a detailed discussion of the fibration structure in the toric context. From our experience with K3 fibrations [6] we expect that many families of toric Calabi–Yau hypersurfaces will have members that admit elliptic fibrations in appropriate regions of the quantum moduli space (or, in technical terms, for a triangulation of the fan that is compatible with the reflexive intersection in the N lattice that provides the fiber).

Toric methods are known to physicists mainly because of the work of Batyrev [7], which appeared in a situation where it had become increasingly clear that complete intersections in products of (weighted) projective spaces were not general enough to grasp phenomena like mirror symmetry [8–11]. In the context of toric geometry, which provides a natural extension of previous constructions, mirror symmetry manifests itself as the elementary duality (or polarity) of polytopes. The classification of toric Calabi–Yau hypersurfaces is equivalent to the enumeration of reflexive polytopes, a problem that can be stated in simple combinatorial terms.

The link between the polytopes that generate the fan defining a toric variety and weights that admit transversal polynomials of appropriate degree for Calabi–Yau hypersurfaces in weighted projective spaces turned out to be very simple, at least in low dimensions: Just take the maximal Newton polytope (MNP), which consists of all exponent vectors of monomials whose degree is equal to the sum of weights. Its dual, which always contains a simplex that encodes the weighted projective space we started with, turns out to be integer for all MNPs in up to four dimensions, implying (by definition) that they are reflexive. In the case of four-dimensional hypersurfaces, however, we will see that only about 20% of our MNPs are reflexive.

Even without transversality weight systems may lead to reflexive polyhedra in the way we just described, and any reflexive polyhedron is a subpolyhedron of an MNP defined by one or several weight systems. This observation is one of the keys to an approach for the classification of reflexive polyhedra [12,13]. In the present context we used it to create large lists of weight systems that lead to reflexive polyhedra. Here the fact that transversality and reflexivity are pracitcally unrelated properties in more than four dimensions becomes even more apparent. Among the 914 164 weight systems we constructed that lead to reflexive polyhedra, less than 1% allow for transversal polynomials!

Fibrations provide a beautiful example of how algebraic structures in a toric variety manifest themselves in terms of linear structures in the N lattice. As we will see, we can identify the fiber as a variety corresponding to some subpolytope Δ^*_{fiber} of Δ^*_{CY} and the base as a variety whose toric description is given in terms of a fan Σ_{fiber} that is a projection of Σ_{CY} along the direction of the sublattice supporting Δ^*_{fiber}. This makes it very easy to look for elliptic fibrations simply by looking for two-dimensional integer subpolytopes containing the interior point. An even simpler approach would start from a multiply weighted space such that one of the weight systems leads to the elliptic fiber. Let us also mention here that our approach is particularly useful for discussing the degeneration of fibers.

In Section 2 we discuss the relation between weighted projective spaces and their toric varieties. We also present formulas and strategies for the calculation of Hodge numbers. In Section 3 we discuss toric fibrations and the toric description of the base manifold. In Section 4 we present our numerical results on weight systems leading to Calabi–Yau 4-folds.

While we were finishing the present work there appeared a preprint [14] that partly overlaps with it.

2. Weighted projective spaces vs. toric varieties

We assume that the reader is familiar with basic notions of toric geometry, such as the definitions of cones and fans (see, e.g., [15] or [16]). We use standard notation, denoting the dual lattices by M and N, their real extensions by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, and the fan in $N_{\mathbb{R}}$ by Σ. To each one-dimensional cone in Σ with primitive generator v_k we assign a homogeneous coordinate [17] z_k, $k = 1, \ldots, N$. From the resulting $\mathbb{C}^N$ we remove the exceptional set

$$Z_\Sigma = \bigcup_I \{(z_1, \ldots, z_N): z_i = 0, \forall i \in I\}, \tag{1}$$

where the union $\bigcup_I$ is taken over all sets $I \subseteq \{1, \ldots, N\}$ for which $\{v_i : i \in I\}$ does not belong to a cone in Σ. Then our toric variety $\mathcal{V}_\Sigma$ is given by the quotient of $\mathbb{C}^N \setminus Z_\Sigma$ by a group which is the product of a finite abelian group and $(\mathbb{C}^*)^{N-n}$ acting by

$$(z_1, \ldots, z_N) \sim (\lambda^{w_j^1} z_1, \ldots, \lambda^{w_j^N} z_N) \quad \text{if} \sum_k w_j^k v_k = 0 \tag{2}$$

($N - n$ of these linear relations are independent). Whenever Σ is simplicial, the corresponding variety $\mathcal{V}_\Sigma$ will have only quotient singularities.

Given a collection of positive integers $(w^1, \ldots, w^{n+1})$, called weights, there are two extreme ways of building a toric variety. In both cases one starts by taking $n+1$ vectors v_i in $\mathbb{R}^n$ such that any n of them are linearly independent and $\sum w^i v_i = 0$. A convenient choice is

$$v_i = e_i, \quad i = 1, \ldots, n, \quad v_{n+1} = -\frac{1}{w^{n+1}} \sum_1^n w^i e_i \tag{3}$$

(or a multiple of these vectors). The M lattice is the lattice freely generated by the v_i.

The weighted projective space $\mathbb{WP}_{(w^1,\ldots,w^{n+1})}$ is the toric variety determined by the unique fan whose one-dimensional cones are $v_1, \ldots, v_{n+1}$. The weights w^i define a grading of monomials by $d(\prod z_i^{a_i}) = \sum w^i a_i$. For the construction of Calabi–Yau hypersurfaces one considers quasi-homogeneous polynomials of degree $d = \sum w^i$. A polynomial W is said to be transverse if the set of equations $\partial W/\partial z_i = 0$ is solved only by $z_i = 0\ \forall i$. This condition ensures that the hypersurface has no singularities in addition to those coming from the singularities of the ambient space. These weight systems were classified for $n \leq 4$ in [9,10].

A sigma model on such a singular variety may be constructed as a particular phase of the low energy limit of an $N = 2$ supersymmetric gauged linear sigma model [18]. A gauged linear sigma model with a single gauge field always contains a Landau–Ginzburg phase. The numbers of chiral primary fields of a given charge in the $N = 2$ superconformal field theory that is the low energy limit of a gauged linear sigma model do not change when going from one phase to another. Therefore we can use Vafa's formulas [19] for charge degeneracies in $N = 2$ superconformal Landau–Ginzburg models to calculate what mathematicians call "physicists' Hodge numbers".

Another, a priori quite different approach, consists in considering the (maximal) Newton polyhedron Δ associated with the most general polynomial W of degree $d = \sum w_i$. This is just the convex hull of the points $(a_1, \ldots, a_{n+1})$ in $\mathbb{Z}^{n+1}$ determined by the exponents occurring in the monomials of W. By construction Δ lies in the hyperplane $\sum w^i a_i = d$ and contains the point $\mathbf{1} = (1, \ldots, 1)$. After changing to n-dimensional integer coordinates, with $\mathbf{1} \to \mathbf{0}$, we may identify the resulting lattice with the M lattice. If the origin $\mathbf{0}$ of M is in the interior of Δ (in this case we say that the weight system has the "interior point property"), the dual polyhedron

$$\Delta^* = \{y \in N_{\mathbb{R}}: \langle y, x\rangle \geq -1 \forall x \in \Delta\} \tag{4}$$

is bounded. If, furthermore, all vertices of Δ^* are in N, Δ is said to be reflexive. The integer generators $v_1, \ldots, v_{n+1}$ of above may be identified with the points dual to the intersection of the planes $x_1 = 0, \ldots, x_{n+1} = 0$ with the hyperplane given by the degree condition (before the change of coordinates). $v_1, \ldots, v_{n+1}$ are points, but not necessarily vertices of Δ^*. If they are, then the coordinate hyperplanes correspond to facets (codim 1 faces) of Δ, i.e. the points of Δ affinely span these hyperplanes. In this case we say that a weight system has the "span property".

It was shown in [13] that transversality always implies the interior point property and that for $n \leq 4$ the interior point property implies reflexivity. The fact that transversality implies reflexivity of Δ for $n \leq 4$ had been checked by computer [20,21]. Note, however, that the proof of [13] also applies to the much larger class of all abelian orbifolds [11] and the MNPs on the respective sublattices that arise by dividing out phase symmetries that still admit transversal polynomials.

Denoting the various sets of weight systems (in obvious notation) by T, I and R, the following relations between the different types hold:

$n = 2, 3$: $T = I = R$,
$n = 4$: $T \subset I = R$,
$n > 4$: $T \subset I$, $R \subset I$, no further relations.

The statement $T = I$ for $n = 2, 3$ is only known due to explicit constructions of the corresponding sets of 3 or 95 weights, respectively [13,22]. In more than two dimensions the span property is independent of whether the weight system belongs to T, I or R.

For a reflexive polyhedron Δ we now consider the fan Σ over some triangulation of the faces of Δ^*. A Calabi–Yau hypersurface in $\mathcal{V}_\Sigma$ is given by the zero locus of

$$p = \sum_{x \in \Delta \cap M} a_x \prod_{k=1}^{N} z_k^{\langle v_k, x\rangle + 1}. \tag{5}$$

If Σ is defined by a maximal triangulation of Δ^*, the generic hypersurface of this type is smooth for $n \leq 4$ [7]. Also by [7], the Hodge numbers h_{11} and $h_{1,n-2}$ are known, and in [23] the remaining Hodge numbers of the type h_{1i} were calculated. For a hypersurface of dimension $n - 1 \geq 3$ these formulas can be summarised as

$$h_{1i} = \delta_{1i}\left(l(\Delta^*) - n - 1 - \sum_{\operatorname{codim}\theta^* = 1} l^*(\theta^*)\right) + \delta_{n-2,i}\left(l(\Delta) - n - 1 - \sum_{\operatorname{codim}\theta = 1} l^*(\theta)\right) + \sum_{\operatorname{codim}\theta^* = i+1} l^*(\theta^*) l^*(\theta) \tag{6}$$

for $1 \leq i \leq n - 2$, where l denotes the number of integer points of a polyhedron and l^* denotes the number of interior integer points of a face. For $n \leq 4$, the generic $(n - 1)$-dimensional Calabi–Yau hypersurface in the family defined by Δ will be smooth and the meaning of these numbers is unambiguous. For $n \geq 5$, the Calabi–Yau variety may have singularities that do not allow a blow-up. In this case we refer the reader to Ref. [23] for a discussion of the precise meaning of the Hodge numbers resulting from Eq. (6).

For Calabi–Yau 4-folds there is a linear relation among the Hodge numbers that has been obtained using index theorems in [14,24]. The same relation can, in fact, be obtained as a simple consequence of a sum rule for charge degeneracies of Ramond ground states that has been derived from modular invariance of the elliptic genus for arbitrary $N = 2$ superconformal field theories [25]:

$$\operatorname{tr}(-)^F J_0^2 = \tfrac{1}{36} c \operatorname{tr}(-)^F = -\tfrac{1}{12} d\chi, \tag{7}$$

where J_0 is the (left-moving) U(1) charge, $c = 3d$ is the central charge and the trace extends over the Ramond ground states (we need to be careful with the sign of the Euler characteristic because the Hodge numbers h_{pq} of the σ model on a Calabi–Yau manifold and the charge degeneracies n_{pq} of Ramond ground states of charge $(Q_L, Q_R) = (p - \frac{1}{2}d, q - \frac{1}{2}d)$ are related by $h_{p,q} = n_{d-p,q}$). For a Calabi–Yau manifold of arbitrary dimension this implies

$$\sum_{p,q=0}^{d} (-)^{p+q} \left(p - \frac{d}{2}\right)^2 h_{p,q} = -\frac{d}{12}\chi. \tag{8}$$

In four dimensions this equation is equivalent to

$$h_{22} = 44 + 4h_{11} - 2h_{12} + 4h_{13}, \tag{9}$$

where we used Poincaré and Hodge duality of the Hodge diamond and omitted the contribution $20h_{02} - 52h_{01}$ on the RHS. that vanishes for toric Calabi–Yau hypersurfaces. For the Euler characterictic of 4-folds we thus find

$$\chi = 6(8 + h_{1,1} - h_{1,2} + h_{1,3}). \tag{10}$$

In two dimensions the above sum rule uniquely determines the Hodge diamond of the K3 surface, whereas it is trivially satisfied (and therefore no so well-known) for CY 3-folds.

As an example for different cases that can occur in different dimensions consider the weight system $(1, \ldots, 1, 2)$. The weighted projective space $\mathbb{WP}_{(1,\ldots,1,2)}$ can be represented by the vectors

$$v_1 = (1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1), v_{n+1} = (-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}) \tag{11}$$

in the lattice N consisting of points in $\mathbb{R}^n$ with coordinates that are either all integer or all half integer. Independently of n, $\mathbb{WP}_{(1,\ldots,1,2)}$ has precisely one pointlike singularity, located at $z_1 = \cdots = z_n = 0$ and determined by the $\mathbb{Z}_2$ quotient $(z_1, \ldots, z_{n+1}) \sim (-z_1, \ldots, -z_n, z_{n+1})$. In terms of toric geometry, this singularity corresponds to the fact that the simplex spanned by $0, v_1, \ldots, v_n$ has twice the volume of the unit simplex. This singularity can always be resolved by blowing up the singular point, i.e. by introducing an exceptional divisor (for a nice description of the blowup of orbifold singularities in a recent review for physicists, see [26]). In terms of toric geometry, this means that we subdivide the cone generated by $v_1, \ldots, v_n$ by adding an extra generator

$$v_{n+2} := (\tfrac{1}{2}, \ldots, \tfrac{1}{2}) = -v_{n+1}. \tag{12}$$

As we will see, however, the fan over Δ^* does not necessarily correspond to this type of desingularisation.

The M lattice can be identified with the integer points in $\mathbb{Z}^n$ such that $\sum x_i = 0 \bmod 2$. For even n, W can be chosen as the Fermat polynomial

$$z_1^{n+2} + \cdots + z_n^{n+2} + z_{n+1}^{(n+2)/2}, \tag{13}$$

so that the maximal Newton polytope Δ is a simplex, whereas for odd n the vertex corresponding to $z_{n+1}^{(n+2)/2}$ is replaced by n vertices corresponding to expressions $z_{n+1}^{(n+1)/2} z_i$ with $i = 1, \ldots, n$. Let us now consider what happens for various values of n:

$n = 2$: The vertices of Δ^* are just v_1, v_2 and v_3. v_4 is in the interior of the edge (facet) $\overline{v_1 v_2}$.

Whether we blow up $\mathbb{WP}_{(1,1,2)}$ by the divisor corresponding to v_4 does not matter because this divisor does not intersect the hypersurface.

$n = 3$: The monomials $z_4^2 z_i$ correspond to a plane in the M lattice whose dual is v_5. Thus $\mathcal{V}_\Sigma$ is the blow-up of $\mathbb{WP}_{(1,1,1,2)}$.

$n = 4$: Once again W is of Fermat type, but now v_6 lies outside Δ^*. The variety $\mathcal{V}_\Sigma = \mathbb{WP}_{(1,1,1,1,2)}$ has the $\mathbb{Z}_2$ singularity, but the generic hypersurface of degree 6 does not intersect it. The blow-up of $\mathbb{WP}_{(1,1,1,1,2)}$ corresponds to a different reflexive polyhedron leading to a hypersurface with different Hodge numbers.

$n = 5$: The monomials $z_6^3 z_i$ correspond to a plane in the M lattice whose dual is the point $\frac{1}{2}v_7 \in N_\mathbb{R}$ which is not in N. Δ^* is no longer reflexive. As there is no Fermat type monomial in z_6, any degree 7 hypersurface in $\mathbb{WP}_{(1,1,1,1,1,2)}$ intersects the singular point $z_1 = \ldots = z_6 = 0$. Vafa's formulas give $h_{11} = 1, h_{12} = 0$ and $h_{13} = 455$. They certainly do not correspond to the blow-up of $\mathbb{WP}_{(1,1,1,1,1,2)}$. Whereas the convex hull $\tilde{\Delta}^*$ of $v_1, \ldots, v_7$ is reflexive and the corresponding variety even smooth, $l(\tilde{\Delta}^*)$ being 8, Eq. (6) tells us that $\tilde{h}_{11} = 8 - 6 = 2$. We note, however, that Eq. (6) can be applied to non-integer polyhedra as well. In the present case the results of inserting Δ, Δ^* into this formula coincide with those of Vafa's formulas. We do not know whether this is always true.

3. Fibrations

The aim of this section is to give a general recipe for identifying fibrations of hypersurfaces of holonomy $SU(n-1)$ in n-dimensional toric varieties where the generic fiber is an $(n'-1)$-dimensional variety of holonomy $SU(n'-1)$. In other words, it will apply to elliptic fibrations of K3 surfaces, CY 3-folds, CY 4-folds, etc., to K3 fibrations of CY k-folds with $k \geq 3$, to 3-fold fibrations of 4-folds, and so on. The main message is that the structures occurring in the fibration are reflected in structures in the N lattice. The fiber, being an algebraic subvariety of the whole space, is encoded by a polyhedron Δ^*_{fiber} which is a subpolyhedron of Δ^*_{CY}, whereas the base, which is a projection of the fibration along the fiber, can be seen by projecting the N lattice along the linear space spanned by Δ^*_{fiber}. The details given in the following are somewhat technical; the reader is advised to check the various steps with some explicit example, e.g. the one given later.

Assume that Δ^* contains a lower-dimensional reflexive subpolyhedron $\Delta^*_{\text{fiber}} = (N_{\text{fiber}})_\mathbb{R} \cap \Delta^*_{\text{CY}}$ with the same interior point. This allows us to define a dual pair of exact sequences

$$0 \to N_{\text{fiber}} \to N_{\text{CY}} \to N_{\text{base}} \to 0 \tag{14}$$

and

$$0 \to M_{\text{base}} \to M_{\text{CY}} \to M_{\text{fiber}} \to 0. \tag{15}$$

Using the same arguments as in [6], we can convince ourselves that the image of Δ_{CY} under $M_{\text{CY}} \to M_{\text{fiber}}$ is dual to Δ^*_{fiber}. Let us also assume that the image Σ_{base} of Σ_{CY} under $\pi : N_{\text{CY}} \to N_{\text{base}}$ defines a fan in N_{base}. This is certainly not true for arbitrary triangulations of Δ^*. Constructing fibrations, one should rather build a fan Σ_{base} from the images of the one-dimensional cones in Σ_{CY} and try to construct a triangulation of Σ_{CY} and thereby of Δ^*_{CY} that is compatible with the projection. It would be interesting to know whether this is always possible whenever the intersction of a reflexive polyhedron with a linear subspace of $N_\mathbb{R}$ is again reflexive.

The set of one-dimensional cones in Σ_{base} is the set of images of one-dimensional cones in Σ_{CY} that do not lie in N_{fiber}. The image of a primitive generator v_i of a cone in Σ_{CY} is the origin or a positive integer multiple of a primitive generator $\tilde{v}_j$ of a one-dimensional cone in Σ_{base}. Thus we can define a matrix r_j^i, most of whose elements are 0, through $\pi v_i = r_i^j \tilde{v}_j$ with $r_i^j \in \mathbb{N}$ if πv_i lies in the one-dimensional cone defined by $\tilde{v}_j$ and $r_i^j = 0$ otherwise. Our base space is the multiply weighted space determined by

$$(\tilde{z}_1, \dots, \tilde{z}_{\tilde{N}}) \sim (\lambda^{\tilde{w}_j^1} \tilde{z}_1, \dots, \lambda^{\tilde{w}_j^{\tilde{N}}} \tilde{z}_{\tilde{N}}), \qquad j = 1, \dots, \tilde{N} - \tilde{n}, \tag{16}$$

where the $\tilde{w}_j^i$ are any integers such that $\sum_i \tilde{w}_j^i \tilde{v}_i = 0$. The projection map from $\mathcal{V}_\Sigma$ (and, as we will see, from the Calabi–Yau hypersurface) to the base is given by

$$\tilde{z}_i = \prod_j z_j^{r_j^i}. \tag{17}$$

This is well defined: $z_j \to \lambda^{w_k^j} z_j$ leads to $\tilde{z}_i \to \lambda^{w_k^j r_j^i} \tilde{z}_i$ which is among the good equivalence relations because applying π to $\sum w_k^j v_j = 0$ gives $\sum w_k^j r_j^i \tilde{v}_i = 0$.

A generic point in the base space will have $\tilde{z}_i \neq 0$ for all i, implying $z_i \neq 0$ for all $v_i \notin \Delta_{\text{fiber}}^*$. The choice of a specific point in $\mathcal{V}_{\Sigma_{\text{base}}}$ and the use of all equivalence relations except for those involving only $v_i \in \Delta_{\text{fiber}}^*$ allows to fix all z_i except for those corresponding to $v_i \in \Delta_{\text{fiber}}^*$. Thus the preimage of a generic point in $\mathcal{V}_{\Sigma_{\text{base}}}$ is indeed a variety in the moduli space determined by Δ_{fiber}^*.

What we have seen so far is just that $\mathcal{V}_\Sigma$ is a fibration over $\mathcal{V}_{\Sigma_{\text{base}}}$ with generic fiber $\mathcal{V}_{\Sigma_{\text{fiber}}}$ (this is actually the statement of an exercise on p. 41 of ref. [15]) and how this fibration structure manifests itself in terms of homogeneous coordinates. Now we also want to see how this can be extended to hypersurfaces. To this end note that if $v_k \in \Delta_{\text{fiber}}^*$ then $\langle v_k, x\rangle$ only depends on the equivalence class $[x] \in M_{\text{fiber}}$ of x under

$$x \sim y \quad \text{if} \quad x - y \in M_{\text{base}}. \tag{18}$$

Thus we may rewrite Eq. (5) as

$$p = \sum_{[x]\in \Delta_{\text{fiber}} \cap M_{\text{fiber}}} a'_{[x]} \prod_{v_k \in \Delta_{\text{fiber}}^*} z_k^{\langle v_k, [x]\rangle + 1} \quad \text{with } a'_{[x]} = \sum_{x \in [x]} a_x \prod_{v_k \notin \Delta_{\text{fiber}}^*} z_k^{\langle v_k, x\rangle + 1}. \tag{19}$$

In each coordinate patch for $\mathcal{V}_{\Sigma_{\text{base}}}$ this is just an equation for the fiber with coefficients that are polynomial functions of coordinates of the base space.

There are two different occasions upon which the fiber may degenerate: The fiber being a hypersurface in $\mathcal{V}_{\Sigma_{\text{fiber}}}$, it can either happen that $\mathcal{V}_{\Sigma_{\text{fiber}}}$ itself degenerates or that the coefficients of the equation determining the fiber hit a singular point in the moduli space. While we do not know any way to read off the occurrence of the second case from the toric data, we can surely see the first case: Whenever a one-dimensional cone (with primitive generator $\tilde{v}_i$) in Σ_{base} is the image of more than one one-dimensional cone in Σ, the fiber becomes reducible over the divisor $\tilde{z}_i = 0$ determined by v_i. Different components of the fiber correspond to different equations $z_j = 0$ with $\pi v_j = r_j^i \tilde{v}_i$. The intersection patterns of the different components of the reducible fibers are crucial for understanding enhanced gauge symmetries [1,26] and deserve further study.

Let us consider as an example the well-known class of Calabi–Yau hypersurfaces (3-folds) of degree $6n + 12$ in $\mathbb{WP}_{(1,1,n,2n+4,3n+6)}$ [27]. If $12/n$ is integer, the corresponding Newton polyhedron will be a simplex and the dual polyhedron Δ^* can be described as the convex hull of the points

$$(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1),\ (-1,-n,-2n-4,-3n-6) \tag{20}$$

in $N \simeq \mathbb{Z}^4$. The elliptic fiber is determined by Δ^*_{fiber} with vertices

$$(0,0,1,0),\ (0,0,0,1),\ (0,0,-2,-3)\ \text{ in }\ N_{\text{fiber}} = N \cap \{x_1 = x_2 = 0\}. \tag{21}$$

Of course it is just the torus given by the Weierstrass equation in $\mathbb{WP}_{(1,2,3)}$. The projection of N to $N_{\text{base}} = N/N_{\text{fiber}}$ is realised as $\pi : (x_1, x_2, x_3, x_4) \to (x_1, x_2)$ ("throwing away the last two coordinates of each point"). Δ^* gets projected to the convex hull of $(1,0)$, $(0,1)$ and $(-1,-n)$. Each of these points provides a one-dimensional cone in Σ_{base}, which clearly is the fan of the Hirzebruch surface $\mathbb{F}^n$ (compare, for example, with [15, p. 7]). All other integer points are of the form $(0,-l)$ with integer $0 \le l \le \frac{1}{2}n$. They correspond to the one-dimensional cone generated by $(0,-1)$ and give examples for non-trivial values of the r_i^j defined above.

Let us get even more specific and consider the case $n = 4$. Neglecting four points in the interiors of facets of Δ^* (the corresponding divisors do not intersect the Calabi–Yau hypersurface and are therefore irrelevant for the present discussion), we can arrange the integer points of Δ^* according to their images

$$\tilde{v}_1 := (1,0),\ \tilde{v}_2 := (0,1),\ \tilde{v}_3 := (-1,-4),\ \tilde{v}_4 := (0,-1),\ \mathbf{0} := (0,0) \tag{22}$$

in Σ_{base}:

$$\begin{aligned}
\pi v &= r\tilde{v}_1 && \text{for } v_1 := (1,0,0,0),\\
\pi v &= r\tilde{v}_2 && \text{for } v_2 := (0,1,0,0),\\
\pi v &= r\tilde{v}_3 && \text{for } v_3 := (-1,-4,-12,-18),\\
\pi v &= r\tilde{v}_4 && \text{for } v_4 := (0,-1,-3,-4),\ \ v_5 := (0,-1,-4,-6),\\
& && \qquad v_6 := (0,-2,-6,-9),\\
\pi v &= \mathbf{0} && \text{for } v_7 := (0,0,1,0),\ \ v_8 := (0,0,0,1),\ \ v_9 := (0,0,-2,-3).
\end{aligned} \tag{23}$$

The projection to the base is given by

$$\tilde{z}_1 = z_1, \quad \tilde{z}_2 = z_2, \quad \tilde{z}_3 = z_3, \quad \tilde{z}_4 = z_4 z_5 z_6^2. \tag{24}$$

There are many linear relations among the v_i. One of them is $2v_7 + 3v_8 + v_9 = 0$, ensuring that

$$(z_1, \ldots, z_6, z_7, z_8, z_9) \sim (z_1, \ldots, z_6, \lambda^2 z_7, \lambda^3 z_8, \lambda z_9). \tag{25}$$

With respect to this relation, p is quasi-homogeneous of degree 6. With linear redefinitions of z_7, z_8, z_9, one can bring p into Weierstrass form

$$p = z_8^2 - z_7^3 + f(z_1, \dots, z_6) z_7 z_9^4 + g(z_1, \dots, z_6) z_9^6. \tag{26}$$

Let us also briefly discuss strategies for finding fibrations. One strategy is to take reflexive polyhedra and look for lower-dimensional reflexive polyhedra that are contained in them. In particular, looking for elliptic fibrations, one might just intersect Δ^* with any two-dimensional plane spanned by integer points of Δ^*. Checking $\binom{N}{2}$ pairs of points w.r.t. whether the plane spanned by them carries a reflexive subpolyhedron is no challenge to present day computer power, even for large numbers of polyhedra.

An even simpler approach for constructing large numbers of fibrations could make use of the following observation: If Δ', Δ'' are reflexive polyhedra in lattices M', M'', respectively, then

$$\Delta := \Delta' \times \Delta'' = \{(x', x''): x' \in \Delta', x'' \in \Delta''\} \tag{27}$$

is also a reflexive polyhedron. Its dual is

$$\Delta^* = \{(\lambda y', (1-\lambda) y''): y' \in \Delta', y'' \in \Delta'', 0 \le \lambda \le 1\} \tag{28}$$

and the set $\Sigma_{(k)}$ of k-dimensional cones in Σ is given by

$$\Sigma_{(k)} = \{(v', v''): v' \in \Sigma'_{(k')}, v'' \in \Sigma''_{(k'')}, k' + k'' = k\}. \tag{29}$$

Of course $\mathcal{V}_\Sigma = \mathcal{V}_{\Sigma'} \times \mathcal{V}_{\Sigma''}$, so $\mathcal{V}_\Sigma$ is trivially a fibration. The equation $p = 0$ defines a Calabi–Yau $(k-1)$-fold which may be interpreted either as a fibration with base $\mathcal{V}_{\Sigma'}$ and generic fiber a hypersurface in $\mathcal{V}_{\Sigma''}$ or as a fibration with base $\mathcal{V}_{\Sigma''}$ and generic fiber a hypersurface in $\mathcal{V}_{\Sigma'}$. In this way one can immediately construct almost three million elliptic fibration 4-folds by combining one of the 16 two-dimensional reflexive polyhedra with reflexive polyhedra coming from one of the 184 026 weight systems of [13].

For applications in the context of F-theory, we are usually interested in elliptic fibrations with sections. As we have seen, toric divisors whose corresponding rays do not project to $\mathbf{0}$ are interpreted as fibers or components of degenerate fibers. Therefore among our toric divisors the only candidates for sections are determined by rays projecting to $\mathbf{0}$, i.e. by points $v \in \Delta^*_{\text{fiber}}$. These points have two interpretations: On the one hand, v determines a divisor in $\mathcal{V}_{\Sigma_{\text{fiber}}}$ and thereby, if it is a vertex of Δ^*_{fiber}, a divisor D_{fiber} in the elliptic fiber, depending on the base point P. In the case of a curve, a divisor is just a formal sum of points on the curve. On the other hand, v may be interpreted as a point in Δ^*, thus determining a divisor D_{CY} in the Calabi–Yau hypersurface (unless v is interior to a facet of Δ^*). Of course the intersection of D_{CY} with the fiber over a generic point P in the base space will give D_{fiber}. Thus v will correspond to a section if D_{fiber} is a single point. This is the case under the following condition: Δ^*_{fiber} being a polygon with v one of its vertices, the dual $v^* \in \Delta_{\text{fiber}} = (\Delta^*_{\text{fiber}})^*$ of v is an edge of Δ_{fiber}. Then D_{fiber} is a single point if v^* has no interior point. Thus our fibration has a section if the polygon Δ_{fiber} has an edge with no interior point.

To conclude this section, we remark that while the conditions we have given for the construction of fibrations are only sufficient and not necessary because of the possibility of non-toric structures, they seem to cover all examples given so far in the context of string dualities.

4. Scans for 4-folds

We performed systematic searches for the two different types of Calabi–Yau 4-fold models described by a single weight system as analyzed in Section 2. In both cases we used improved versions of the same basic strategy, namely to check for all partitions $w^1, \dots, w^{n+1}$ of $d = 6, 7, \dots$ whether they allow for transverse polynomials or reflexive polyhedra.

For the transverse weights, the preselection that we imposed to reduce the number of partitions is, in the language of [28], "compatibility with at most one unresolved pointer to an unknown weight". With this method we obtained the complete list of 525 572 weight systems of degree up to 4000. Based on the statistics given in Table 1 we cannot give an estimate of the total number, since $2 \times 10^5/d$ is a good approximation to the average number of transversal weights per degree, which would lead to a divergent sum. It is well known, however, that this set is finite [28]. A complete enumeration is impossible with our present approach (this can be inferred from the rate at which our program slows down with growing d and the fact that the degrees of the 3462 Fermat weights range up to $d = 326\,3442$), but the numbers in Table 1 do not seem to exclude the possibility of a complete classification along the lines of Refs. [9,10].

The situation is quite different for 4-folds coming from reflexive weights, which are far more numerous. Here we are limited by disk space rather than by calculation time. We have the complete list of 914 164 weights with degree $d \leq 150$, only 6918 of which are transversal. Together with the weights that are both transversal and reflexive we have thus

Table 1

d	Trans.	Ref.	d	Trans.	Ref.	d	Trans.	Ref.	d	Trans.	Ref.
100	11 798	3578	1100	18 333	3917	2100	11 328	2292	3100	6765	1208
200	28 457	6685	1200	16 351	3261	2200	9694	1773	3200	7449	1369
300	31 075	6716	1300	14 427	3045	2300	8944	1627	3300	6811	1300
400	29 229	6163	1400	15 334	3196	2400	10 807	2314	3400	6476	1282
500	26 792	5798	1500	12 907	2450	2500	7385	1352	3500	6209	1230
600	26 578	5649	1600	13 570	2764	2600	9190	1897	3600	6597	1329
700	22 367	4665	1700	12 432	2454	2700	8470	1700	3700	6061	1218
800	22 139	4725	1800	11 594	2400	2800	8134	1618	3800	6288	1249
900	20 704	4478	1900	11 030	2118	2900	7975	1523	3900	5394	1046
1000	17 475	3605	2000	10 273	1961	3000	6827	1248	4000	5903	1105
	236 614	52 062		136 251	27 566		88 754	17 344		63 953	12 336

There are 109 308 reflexive weights among the 525 572 transversal weights with $d \leq 4000$

Table 2

d	R	RT	T	d	R	RT	T	d	R	RT	T
10	9	8	11	60	11 489	490	1480	110	84 095	509	2235
20	164	63	109	70	19 046	375	1397	120	114 038	955	3482
30	835	209	422	80	30 747	586	2030	130	137 806	456	2005
40	2485	252	684	90	45 815	744	2529	140	178 688	656	2845
50	5724	356	1093	100	62 779	495	2043	150	220 331	764	3070
$\sum$	9217	888	2319	$\sum$	169 876	2690	9479	$\sum$	734 958	3340	13 637

There are 914 051 reflexive, 6918 reflexive and transversal, and 25 435 general transversal weights with $d \leq 150$.

Table 3
Negative values of the Euler number (the value $\chi = -30$ only occurs for non-reflexive weights)

−6	−12	−18	−24	−30	−36	−42	−48		−60	−66	−72
	−84	−90	−96				−120		−132	−138	−144
			−168		−180		−192	−198			
			−240		−252						

Table 4
Numbers of weights with small $1 \leq h_{11} \leq 5$ or $1 \leq h_{13} \leq 5$ in the list of all reflexive weights with $d \leq 150$, and reflexive (RT) or general (T) transversal weights with $d \leq 4000$

h_{11}	R_{150}	RT_{4000}	T_{4000}	h_{13}	R_{150}	RT_{4000}	T_{4000}
1	8	8	33	1	–	6	33
2	33	27	106	2	–	24	132
3	101	66	255	3	–	44	196
4	168	88	411	4	–	48	304
5	267	111	508	5	–	66	354
6	501	183	800	6	–	133	533
7	617	158	789	7	5	95	486

accumulated more than 10^6 reflexive weights. Various sublists, as well as the files with the complete results are available on the internet. [2]

A central goal of our computer studies was to investigate the relation between transversality and reflexivity of the maximal Newton polyhedra. Whereas transversality always implies reflexivity in up to four dimensions, it turned out that only about 20% of the five-dimensional polyhedra defined by our transversal weights are reflexive. This number shows only little dependence (in the form of a slight decrease) on the degree. Since the gauged linear σ models based on these weights have a Landau–Ginzburg phase we could use Vafa's formulas to compute all charge degeneracies of Ramond ground states. For the 109 308 reflexive weights in this class we could thus check the coincidence of these numbers with Batyrev's result for the cohomology of the Calabi–Yau hypersurfaces in the toric varieties

[2] The URL is http://tph.tuwien.ac.at/~kreuzer/CY.

Table 5
Weights for 4-folds with negative Euler number and $d \leq 150$

d	w_1	w_2	w_3	w_4	w_5	w_6	P	V	$\overline{P}$	$\overline{V}$	h_{11}	h_{12}	h_{13}	h_{22}	χ
70	7	7	7	13	17	19	84	8	38	7	26	108	72	220	−12
77	7	7	7	17	19	20	93	8	38	7	26	135	81	202	−120
80	5	5	5	14	22	29	199	8	42	7	24	210	177	428	−6
80	5	5	5	16	18	31	199	8	42	7	24	210	177	428	−6
80	10	13	13	13	15	16	38	6	48	6	38	96	26	108	−144
84	7	7	7	12	18	33	133	6	38	6	22	165	111	246	−144
84	11	11	11	12	18	21	43	6	38	6	28	91	31	98	−144
85	5	5	5	16	23	31	214	8	42	7	24	240	192	428	−96
85	5	5	5	17	21	32	214	8	42	7	24	240	192	428	−96
88	8	8	8	19	22	23	93	8	38	7	26	135	81	202	−120
88	8	11	12	19	19	19	44	9	70	8	63	108	29	196	−48
90	5	5	5	18	24	33	231	6	42	6	22	272	209	424	−198
90	7	7	7	19	20	30	84	8	38	7	26	108	72	220	−12
90	9	9	9	10	16	37	113	6	39	6	22	144	91	208	−138
90	9	9	9	13	20	30	84	8	38	7	26	108	72	220	−12
90	9	9	9	14	19	30	84	8	38	7	26	108	72	220	−12
90	9	9	9	17	22	24	84	8	38	7	26	108	72	220	−12
90	9	10	14	19	19	19	48	6	75	6	65	108	33	220	−12
90	9	12	13	13	13	30	55	7	38	7	27	92	42	136	−90
90	11	11	11	15	18	24	42	6	42	6	30	101	30	82	−198
91	11	13	13	13	16	25	55	9	34	8	23	90	43	128	−96
92	7	7	7	22	23	26	84	8	38	7	26	108	72	220	−12
95	10	13	15	19	19	19	42	10	37	9	30	72	30	140	−24
96	7	7	7	19	24	32	93	8	38	7	26	135	81	202	−120
96	7	12	19	19	19	20	42	10	84	9	75	126	27	200	−96
96	8	11	19	19	19	20	42	9	76	8	69	108	27	212	−24
98	7	7	14	20	24	26	79	8	39	7	27	108	69	212	−24
99	7	7	7	22	23	33	93	8	38	7	26	135	81	202	−120
99	9	9	9	17	22	33	93	8	38	7	26	135	81	202	−120
99	9	9	9	19	20	33	93	8	38	7	26	135	81	202	−120
99	9	9	9	19	23	30	93	8	38	7	26	135	81	202	−120
99	9	10	19	19	19	23	41	9	91	8	81	135	26	202	−120
99	9	11	11	11	15	42	108	6	42	6	24	141	85	198	−144
99	9	11	19	20	20	20	41	9	91	8	81	135	26	202	−120
99	9	14	14	14	15	33	54	6	42	6	29	102	41	120	−144
100	7	7	7	23	25	31	93	8	38	7	26	135	81	202	−120
102	6	6	6	17	32	35	214	8	42	7	24	240	192	428	−96
102	9	12	17	17	17	30	56	8	34	8	25	82	43	152	−36
102	12	17	17	17	18	21	44	7	34	7	26	81	32	114	−90
104	8	13	20	21	21	21	39	6	93	6	83	147	24	178	−192
105	7	7	14	24	26	27	85	9	42	10	30	126	74	208	−84
105	7	15	17	22	22	22	45	12	84	9	74	126	30	208	−84
105	10	15	17	21	21	21	42	9	37	8	30	72	30	140	−24
105	11	15	15	15	21	28	55	9	34	8	23	90	43	128	−96
108	7	7	7	24	27	36	104	6	38	6	23	165	92	174	−252
110	5	5	10	22	26	42	184	8	43	7	25	200	165	404	−12
110	9	15	20	22	22	22	44	9	47	8	36	90	32	136	−84
110	10	11	21	21	21	26	41	9	91	8	81	135	26	202	−120
110	10	15	19	22	22	22	42	9	37	8	30	72	30	140	−24
110	11	11	16	20	22	30	52	9	34	8	24	80	42	148	−36

Table 5
Continued

d	w_1	w_2	w_3	w_4	w_5	w_6	P	V	$\overline{P}$	$\overline{V}$	h_{11}	h_{12}	h_{13}	h_{22}	χ
112	7	7	14	16	24	44	119	6	39	6	23	147	100	242	−96
112	7	16	20	23	23	23	42	9	92	8	81	144	27	188	−168
112	8	13	21	21	21	28	42	9	76	8	69	108	27	212	−24
112	11	11	16	22	24	28	39	6	39	6	29	82	29	112	−96
114	6	14	19	25	25	25	53	10	88	9	77	135	38	234	−72
114	9	15	19	19	19	33	55	7	38	7	27	92	42	136	−90
114	15	18	19	19	19	24	43	7	38	7	28	91	31	98	−144
115	5	5	10	22	31	42	194	9	45	9	27	220	174	408	−66
115	5	5	10	23	29	43	194	9	45	9	27	220	174	408	−66
115	10	15	21	23	23	23	42	9	37	8	30	72	30	140	−24
120	5	5	10	24	32	44	206	6	43	6	23	242	187	400	−144
120	7	7	14	22	30	40	79	8	39	7	27	108	69	212	−24
120	8	8	16	27	30	31	85	9	42	10	30	126	74	208	−84
120	8	14	15	15	15	53	111	6	49	6	29	147	86	210	−144
120	8	15	23	23	23	28	39	6	93	6	83	147	24	178	−192
120	10	17	17	18	24	34	39	7	57	7	47	92	29	164	−48
120	11	11	20	22	24	32	38	6	43	6	31	91	28	98	−144
120	12	13	13	16	26	40	50	7	39	7	28	83	39	146	−48
121	10	11	21	21	21	37	41	9	91	8	81	135	26	202	−120
125	4	13	25	25	25	33	87	10	72	8	56	144	68	252	−72
126	9	9	18	20	28	42	79	8	39	7	27	108	69	212	−24
126	9	9	18	22	26	42	79	8	39	7	27	108	69	212	−24
126	11	11	11	14	16	63	113	6	39	6	22	144	91	208	−138
126	13	13	18	21	26	35	34	8	55	9	43	90	23	128	−96
128	7	7	14	30	32	38	79	8	39	7	27	108	69	212	−24
130	7	7	7	18	26	65	199	8	42	7	24	210	177	428	−6
130	10	10	13	20	24	53	100	9	–	9	26	120	80	228	−36
132	7	7	14	27	33	44	85	9	42	10	30	126	74	208	−84
132	8	8	8	31	33	44	93	8	38	7	26	135	81	202	−120
132	11	11	12	20	22	56	97	6	43	6	25	126	77	200	−96
132	11	12	23	23	23	40	38	6	104	6	92	165	23	174	−252
132	12	12	12	19	33	44	93	8	38	7	26	135	81	202	−120
132	12	12	12	23	33	40	93	8	38	7	26	135	81	202	−120
132	12	13	13	13	15	66	108	6	42	6	24	141	85	198	−144
132	12	15	22	22	22	39	54	6	42	6	29	102	41	120	−144
132	12	17	17	22	30	34	34	6	62	6	52	100	24	148	−96
133	7	7	21	31	33	34	90	8	49	7	37	135	78	234	−72
134	7	7	7	22	24	67	199	8	42	7	24	210	177	428	−6
135	4	15	27	27	27	35	87	10	72	8	56	144	68	252	−72
135	7	7	14	30	32	45	85	9	42	10	30	126	74	208	−84
135	9	9	18	26	31	42	85	9	42	10	30	126	74	208	−84
135	15	19	20	27	27	27	38	6	48	6	38	96	26	108	−144
136	7	7	14	31	34	43	85	9	42	10	30	126	74	208	−84
136	8	8	8	21	23	68	214	8	42	7	24	240	192	428	−96
136	16	17	17	24	28	34	40	7	35	7	27	73	30	126	−48
138	6	6	12	23	44	47	194	9	45	9	27	220	174	408	−66
138	6	22	23	29	29	29	49	6	104	6	91	165	34	214	−192
140	5	13	28	28	28	38	83	6	83	6	64	168	64	220	−192
140	7	7	20	21	30	55	123	7	49	7	32	150	102	280	−48
140	10	19	19	26	28	38	38	6	64	6	52	104	28	156	−96

Table 5
Continued

d	w_1	w_2	w_3	w_4	w_5	w_6	P	V	$\overline{P}$	$\overline{V}$	h_{11}	h_{12}	h_{13}	h_{22}	χ
140	11	11	20	30	33	35	40	7	49	7	39	83	28	146	−48
140	20	20	20	21	24	35	51	8	–	7	27	90	39	128	−96
142	7	7	7	23	27	71	214	8	42	7	24	240	192	428	−96
143	10	13	13	19	26	62	98	10	–	10	31	120	76	232	−30
144	7	7	14	32	36	48	93	6	39	6	24	147	83	178	−192
144	9	9	18	28	32	48	93	6	39	6	24	147	83	178	−192
144	9	16	16	21	34	48	57	6	63	6	48	114	42	176	−96
144	16	16	18	21	32	41	48	8	–	9	27	84	37	132	−72
144	16	17	17	18	34	42	33	6	69	6	57	112	23	140	−144
145	5	5	15	28	39	53	204	8	54	7	35	232	183	452	−36
145	5	5	15	29	37	54	204	8	54	7	35	232	183	452	−36
145	5	14	29	29	29	39	83	6	83	6	64	168	64	220	−192
150	7	7	7	24	30	75	231	6	42	6	22	272	209	424	−198
150	10	14	17	17	17	75	111	6	49	6	29	147	86	210	−144
150	11	11	25	30	33	40	39	7	54	7	42	92	27	136	−90
150	13	13	15	20	39	50	51	8	49	8	38	85	38	178	−6
150	21	24	25	25	25	30	42	6	42	6	30	101	30	82	−198

Degrees of reflexive *and* transversal weights are in boldface, V and $\overline{V}$ denote the numbers of vertices of Δ and Δ^*, and non-reflexive weights have no entry for the number $\overline{P}$ of lattice points of Δ^*.

defined by the reflexive polytopes. The subtle connections between the two types of Hodge numbers are discussed in [23]. Going from the weighted projective space to the variety defined by a maximal triangulation of the reflexive polyhedron we do not resolve all of the singularities of the ambient space. As we saw in our example at the end of Section 2, even in cases when there exists an "obvious" way of blowing up the embedding variety, this need not be the one leading to the same Hodge numbers.

After the general statistics of transversal and reflexive weights is Tables 1 and 2 we provide lists of weights that may be of particular interest because of small Hodge numbers h_{11} or h_{13}, or because of a negative Euler number, which is desirable in the context of SUSY breaking [4]. In Table 3 we give the possible negative Euler number that arise in our lists. The value $\chi = -30$ only occurs for non-reflexive weights, and our smallest value $\chi = -252$ (which is not divisible by 24) occurs at degree 108 in the reflexive case and only at degree 484 in the transversal case. In Table 4 we give the numbers of weights of various types that we find with $h_{11} \leq 7$ or $h_{13} \leq 7$. In Table 5 we give all 30 transversal and 113 reflexive weights with $\chi < 0$ and $d \leq 150$; there is an overlap of 26 weights in this table that are both reflexive and transversal. Up to degree 4000 we found 174 more transversal weights with negative Euler number, so that altogether we have 291 weights with $\chi < 0$ (the smallest values of h_{11} and h_{13} in this list are both 22).

In Tables 6 and 7 we list all our weights with $h_{11} = 1$ or $h_{13} = 1$; Fermat weights all have $\chi > 0$ and do not contribute to any of our tables of special weights, except for 8 weights with $6 \leq d \leq 18$ that give $h_{11} = 1$. Some "first occurrences" are:

- The first non-reflexive transversal weight system is (1, 1, 1, 1, 1, 2).
- The first non-transversal reflexive weight system is (1, 1, 1, 1, 1, 3).

Table 6
Weights for 4-folds with $h_{11} = 1$ (for reflexive Δ this implies $V = \overline{V} = 6$)

d	w_1	w_2	w_3	w_4	w_5	w_6	P	V	$\overline{P}$	$\overline{V}$	h_{11}	h_{12}	h_{13}	h_{22}	χ
6	1	1	1	1	1	1	462	6	7	6	1	0	426	1752	2610
7	1	1	1	1	1	2	496	10	–	7	1	0	455	1868	2784
8	1	1	1	1	2	2	483	6	7	6	1	0	443	1820	2712
9	1	1	1	1	2	3	575	10	–	7	1	0	523	2140	3192
10	1	1	1	1	1	5	1128	6	8	6	1	0	976	3952	5910
10	1	1	1	2	2	3	489	10	–	7	1	0	447	1836	2736
12	1	1	1	1	2	6	1167	6	8	6	1	0	1009	4084	6108
12	1	1	1	2	3	4	603	6	7	6	1	0	547	2236	3336
12	1	1	2	2	3	3	407	6	7	6	1	0	373	1540	2292
14	1	1	1	2	2	7	1081	6	8	6	1	0	935	3788	5664
15	1	1	2	3	3	5	492	10	–	7	1	0	447	1836	2736
16	1	1	1	2	3	8	1226	9	–	7	1	0	1059	4284	6408
16	1	1	2	3	4	5	509	13	–	9	1	0	463	1900	2832
18	1	1	2	2	3	9	984	6	8	6	1	0	851	3452	5160
18	1	2	3	3	4	5	309	14	–	9	1	0	283	1180	1752
20	1	2	3	4	5	5	314	10	–	7	1	0	287	1196	1776
21	1	1	3	4	5	7	564	12	–	9	1	0	511	2092	3120
21	1	2	3	3	5	7	378	13	–	8	1	0	343	1420	2112
22	1	1	2	3	4	11	1095	12	–	8	1	0	946	3832	5730
22	1	2	3	4	5	7	358	20	–	13	1	0	326	1352	2010
24	2	3	3	4	5	7	187	13	–	9	1	0	171	732	1080
26	1	2	2	3	5	13	855	12	–	8	1	0	739	3004	4488
28	1	1	3	4	5	14	1148	12	–	8	1	0	991	4012	6000
28	1	3	4	5	7	8	300	17	–	13	1	0	273	1140	1692
30	1	2	3	4	5	15	759	9	–	7	1	0	656	2672	3990
30	2	3	4	5	7	9	189	16	–	12	1	0	172	736	1086
36	1	2	3	5	7	18	896	13	–	10	1	0	773	3140	4692
40	1	3	4	5	7	20	685	14	–	10	1	0	591	2412	3600
42	2	3	4	5	7	21	418	13	–	9	1	0	361	1492	2220
42	4	5	6	7	9	11	93	19	–	16	1	0	84	384	558
46	1	4	5	6	7	23	599	19	–	13	1	0	517	2116	3156
50	2	3	5	7	8	25	419	17	–	12	1	0	361	1492	2220
60	3	4	5	7	11	30	316	14	–	11	1	0	271	1132	1680

- The first degree that does not admit any weight that is transversal and reflexive is 11.
- The first degree for which no transversal weight system exists is 1733 (there are only 29 such degrees d in the range $d \leq 4000$).

When looking for models with very specific features it may be useful to go beyond the class of models considered here. In particular if we are interested in elliptic fibrations it will be more ecomonic to generate reflexive polytopes in terms of combined weight systems [12] since the complete set of relevant weights is already known [13] and the fibrations structure is encoded (and can be pre-selected) in a rather simple and explicit way. Note also that the three million elliptic fibrations that were mentioned at the end of the last section are known to be all connected in a web with singular transitions that respect the fibration structure because the same is true for the polytopes of which we are taking direct products [29].

Table 7
Weights for 4-folds with $h_{13} = 1$ (in this case all polytopes are simplices)

d	w_1	w_2	w_3	w_4	w_5	w_6	P	V	$\overline{P}$	$\overline{V}$	h_{11}	h_{12}	h_{13}	h_{22}	χ
2415	105	230	279	462	534	805	7	6	–	6	273	0	1	1140	1692
2484	96	265	276	597	621	629	7	6	–	6	273	0	1	1140	1692
2520	180	215	336	364	585	840	7	6	–	6	283	0	1	1180	1752
2565	105	194	410	431	570	855	7	6	–	6	326	0	1	1352	2010
2700	123	240	263	540	675	859	7	6	–	6	273	0	1	1140	1692
2700	270	300	369	475	486	800	7	6	407	6	373	0	1	1540	2292
2777	335	395	397	407	476	767	7	6	–	6	455	0	1	1868	2784
3024	268	336	384	467	689	880	7	6	–	6	447	0	1	1836	2736
3108	123	279	296	597	777	1036	7	6	–	6	273	0	1	1140	1692
3120	240	260	481	576	715	848	7	6	–	6	447	0	1	1836	2736
3125	434	500	520	521	525	625	7	6	462	6	426	0	1	1752	2610
3216	171	237	536	609	670	993	7	6	–	6	463	0	1	1900	2832
3234	385	390	474	539	552	894	7	6	–	6	455	0	1	1868	2784
3240	391	407	450	540	558	894	7	6	–	6	455	0	1	1868	2784
3240	396	397	461	474	648	864	7	6	–	6	455	0	1	1868	2784
3241	391	461	463	475	556	895	7	6	–	6	455	0	1	1868	2784
3276	396	403	455	546	576	900	7	6	–	6	455	0	1	1868	2784
3360	240	260	517	672	775	896	7	6	–	6	447	0	1	1836	2736
3432	184	312	429	655	812	1040	7	6	–	6	463	0	1	1900	2832
3456	150	415	432	551	756	1152	7	6	–	6	343	0	1	1420	2112
3528	387	432	504	588	637	980	7	6	483	6	443	0	1	1820	2712
3528	432	441	504	516	631	1004	7	6	483	6	443	0	1	1820	2712
3582	350	404	454	597	782	995	7	6	–	6	523	0	1	2140	3192
3584	392	448	456	467	782	1039	7	6	–	6	523	0	1	2140	3192
3600	352	450	464	525	784	1025	7	6	–	6	523	0	1	2140	3192
3696	184	336	439	693	924	1120	7	6	–	6	463	0	1	1900	2832
3750	521	600	624	625	630	750	7	6	462	6	426	0	1	1752	2610
3780	268	420	480	567	945	1100	7	6	–	6	447	0	1	1836	2736
3780	335	420	480	689	756	1100	7	6	–	6	447	0	1	1836	2736
3780	391	525	540	630	651	1043	7	6	–	6	455	0	1	1868	2784
3780	461	462	540	553	756	1008	7	6	–	6	455	0	1	1868	2784
3888	288	400	436	605	863	1296	7	6	–	6	447	0	1	1836	2736
3960	360	396	400	890	891	1023	7	6	603	6	547	0	1	2236	3336

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